AREAS

EXAMPLE 5 (Integration with care)

Consider the function $y = x^2 - 4x$ on $[0, 4]$.

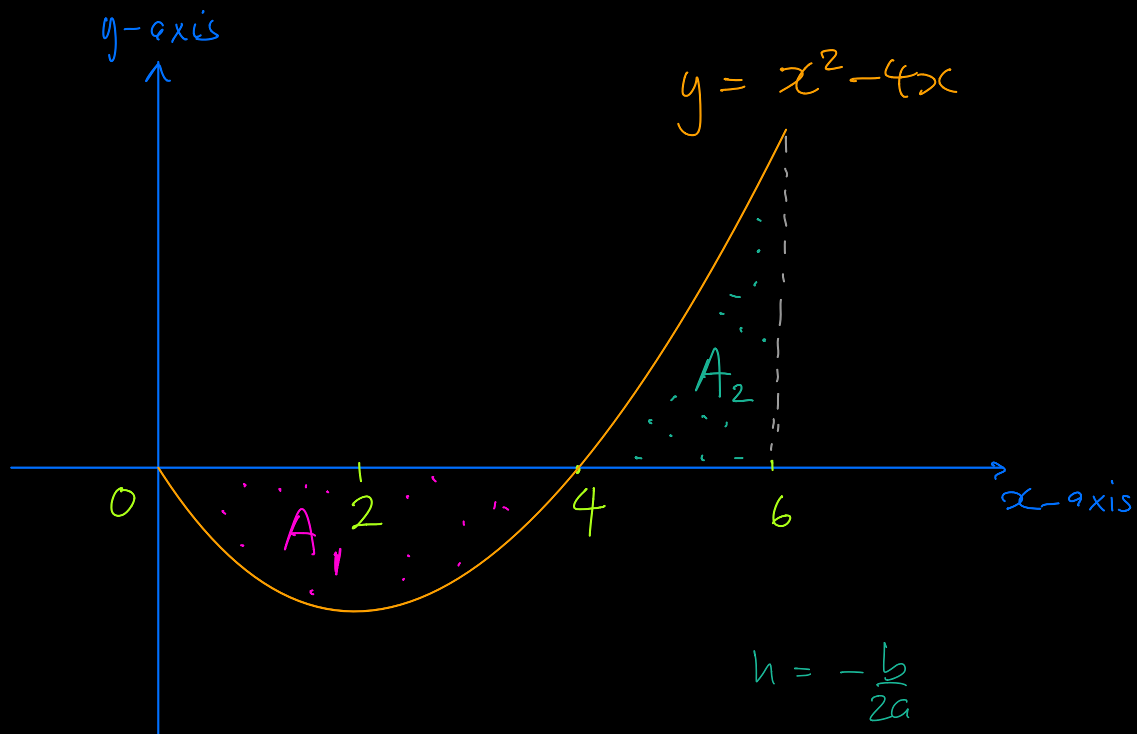
(a) Sketch the graph of $y(x) = x^2 - 4x$.

(b) Find the area trapped between the curve and the x-axis.

Solⁿ: 5(a)

$$y = x^2 - 4x$$
$$= x(x-4)$$

$\therefore$ roots of $y(x) = 0$ are at $x = \{0, 4\}$.



$$h = -\frac{b}{2a}$$
$$= \frac{1}{2}(\alpha + \beta)$$
$$= \frac{1}{2}(0+4) = 2$$

Sol^n: 5(b) The required area is $A_T = |A_1| + A_2$

Now, $A_1 = \int_0^4 y \, dx$

$$= \int_0^4 x^2 - 4x \, dx$$

$$= \left[\frac{x^3}{3} - 2x^2 \right]_0^4$$

$$= \left(\frac{64}{3} - 32 \right) - (0 - 0)$$

$$= 21\tfrac{1}{3} - 32$$

$$= -10\tfrac{2}{3}$$

$= 10\frac{2}{3}$ sq. units below the x-axis

Similarly,

$$A_2 = \int_4^6 y \, dx$$

$$= \int_4^6 x^2 - 4x \, dx$$

$$\therefore \quad A_2 = \left[\frac{x^3}{3} - 2x^2 \right]_4^6$$

$$= (72 - 72) - \left(\frac{64}{3} - 32\right)$$

$$= 0 - 21\tfrac{1}{3} + 32$$

$$= 10\tfrac{2}{3} \text{ sq. units above the } x\text{-axis.}$$

Hence, the final result is

$$A_T = |A_1| + A_2$$

$$= 10\tfrac{2}{3} + 10\tfrac{2}{3}$$

$$= 20\tfrac{4}{3}$$

$$= 21\tfrac{1}{3}$$ sq. units between the constrained* curve and the x-axis

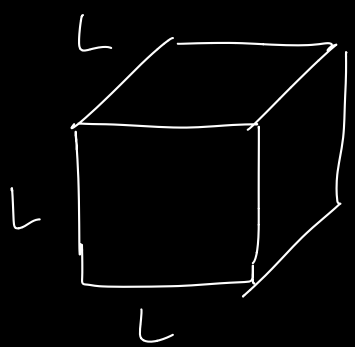
* The constraint is $0 \le x \le 6$ (i.e. the limits)

SOLIDS

BRIEF

Review of Volumes & Surface Areas

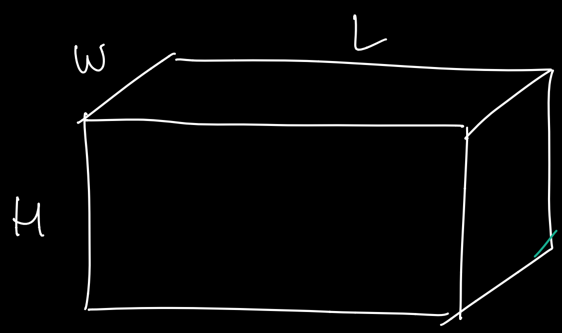
CUBE



$V = L^3$

$A = 6L^2$

CUBOID

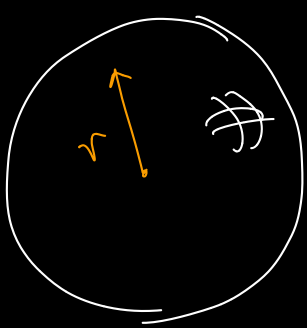


$V = LWH$

$A = 2\left(HW + LW + LH\right)$

* The cube and the cuboid are special prisms

SPHERE



$V = \frac{4}{3}\pi r^3$

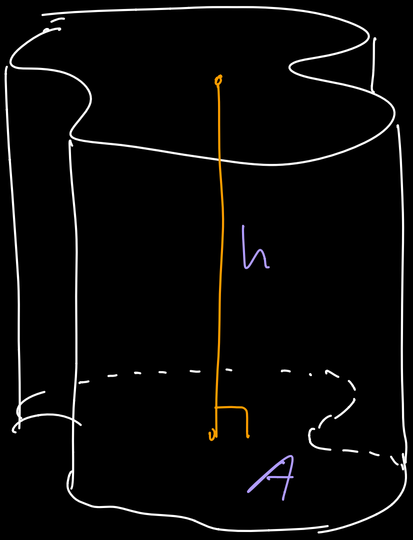
$A = \frac{dV}{dr}$ (unique!)

$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{4}{3}\pi \times 3r^2$

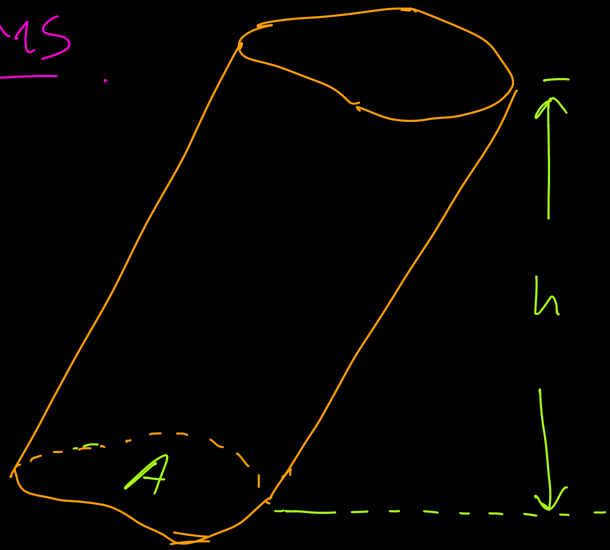
$= 4\pi r^2$

PRISMS



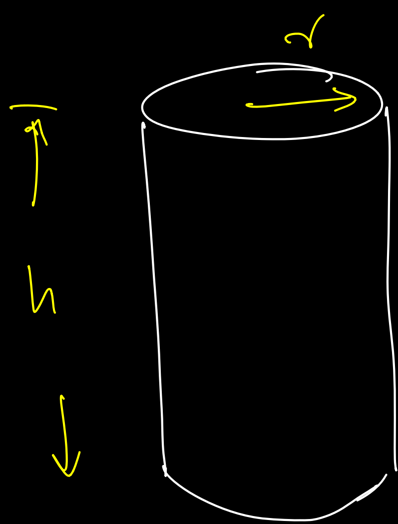
right prism

$V = Ah$



non-right prism

$V = Ah$



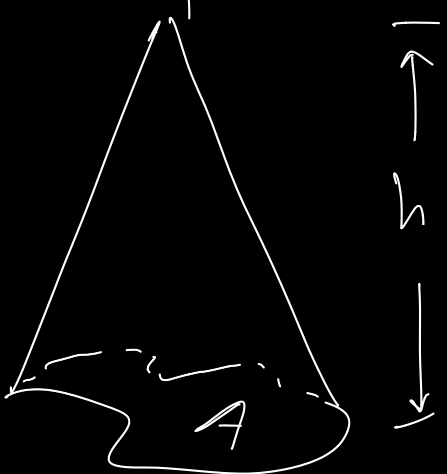
$V = \pi r^2 h$

$A = 2\pi r^2 + 2\pi r h$

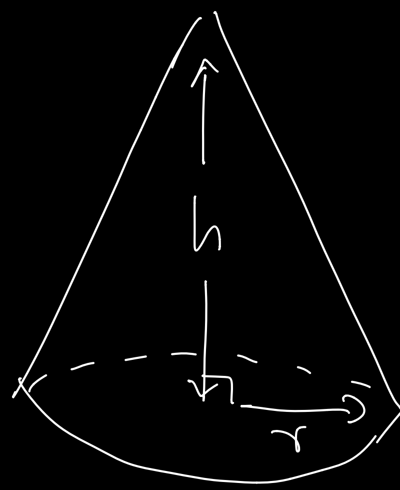
(closed)

PYRAMIDS

apex



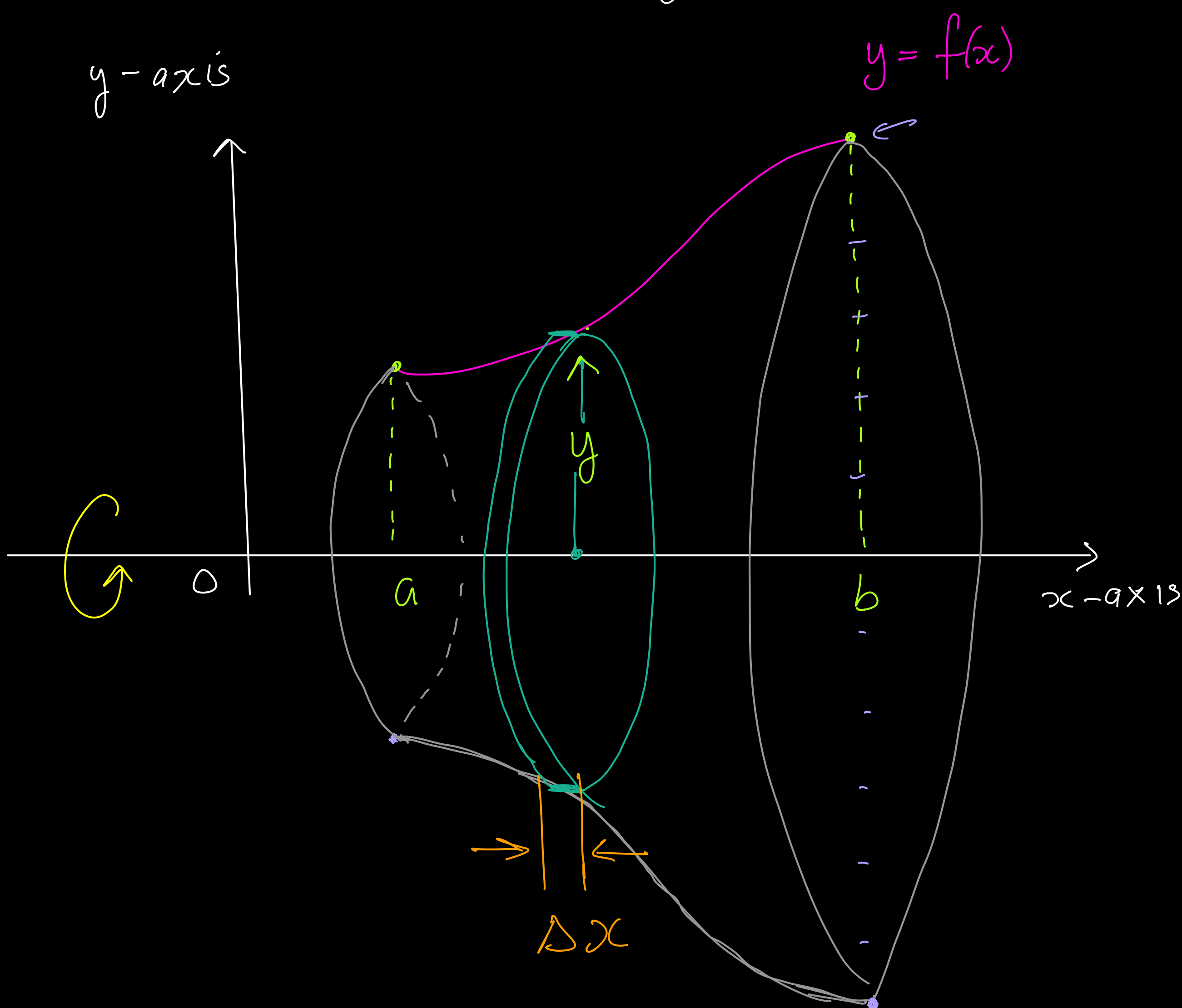
$V = \frac{1}{3} A h$



$V = \frac{1}{3} \pi r^2 h$

SOLIDS OF REVOLUTION VOLUMES

Consider the rotation of $y = f(x)$ about the x-axis.



The element of volume is

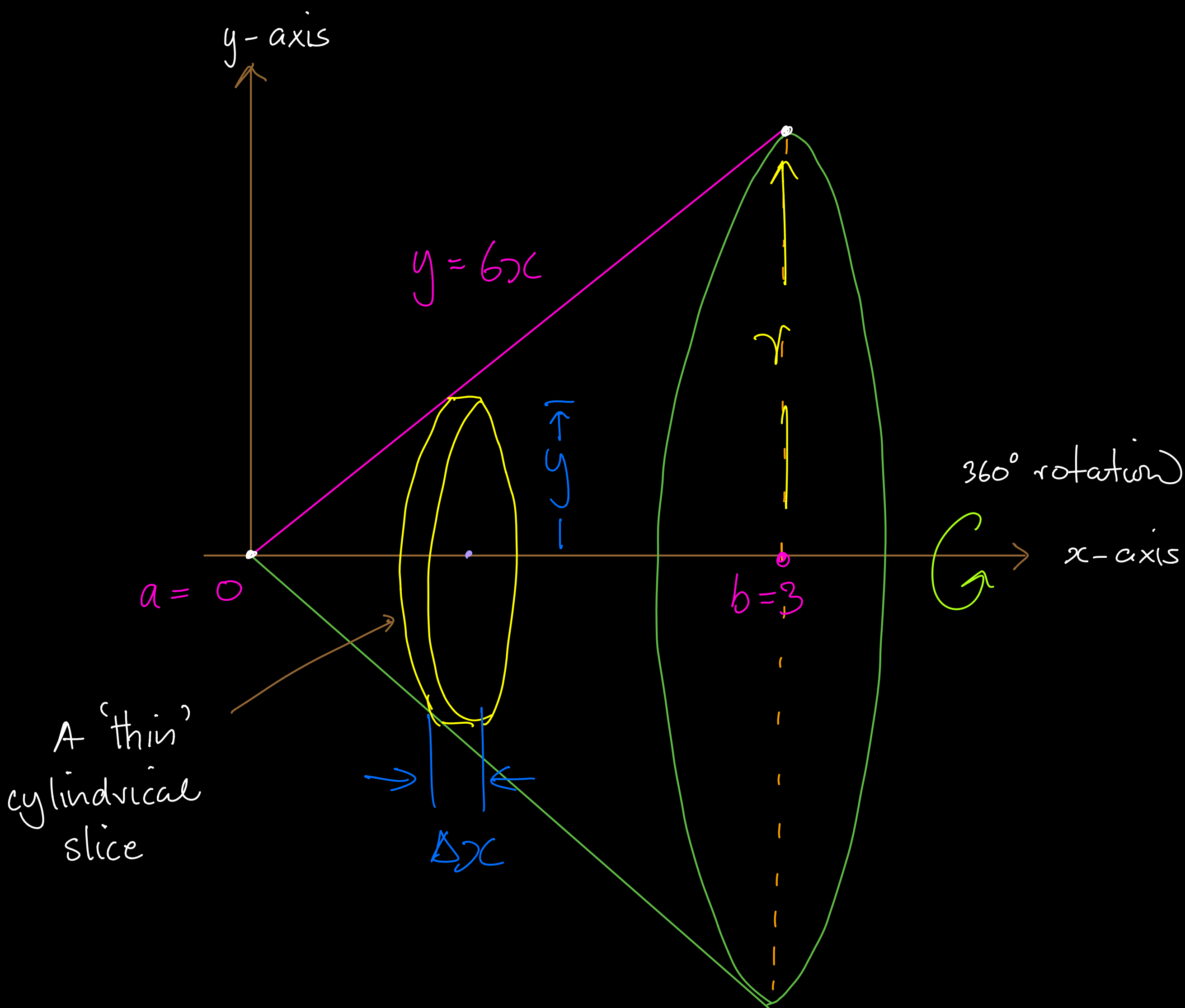
$$\Delta V = \pi y^2 \Delta x$$

$\therefore$ In the limit as $\delta x \to dx$ and $\Delta V \to dV$, using integration, the total volume is

$$V = \int_a^b \pi y^2 \, dx, \quad 360^\circ \text{ about the } x\text{-axis}$$

EXAMPLE 6 Find the volume of revolution when $y = 6x$ is rotated 360° about the x-axis in the 1^{st} Quadrant up to $x = 3$.

Solⁿ: 6 A sketch of the volume formed by the specified mode of rotation is as below.



Element of volume is

$$\Delta V = \pi y^2 \Delta x$$

$\therefore$ In the limit, using integration

$$V = \int_a^b \pi y^2 \, dx$$

$$= \int_0^3 \pi (6x)^2 \, dx$$

$$= \int_0^3 36\pi x^2 \, dx$$

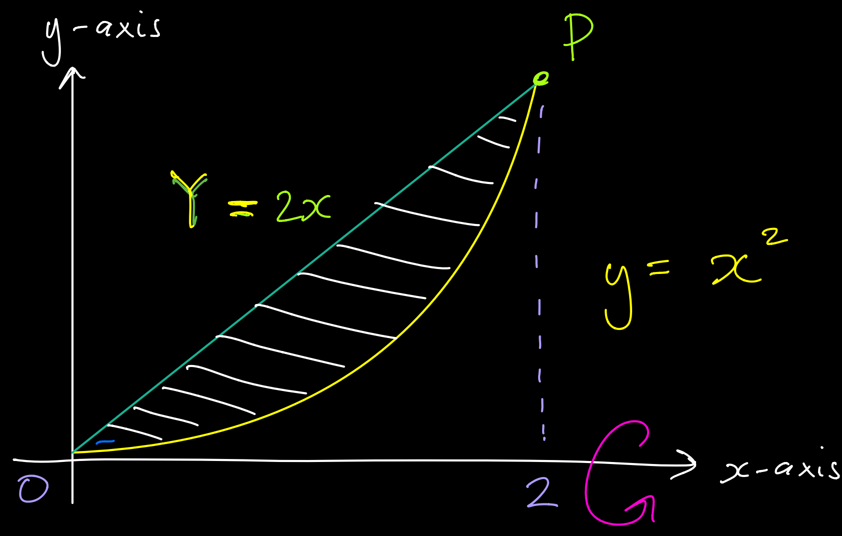
$$= \left[12\pi x^3 \right]_0^3$$

$$= 12\pi (27 - 0)$$

$$= 324\pi \quad \text{cubic units}$$

EXAMPLE 7

Consider the graph shown.



Find (a) y_P

(b) Volume formed by rotating the shaded region around the x-axis once.

Solⁿ 7(a) $y_P = Y(2)$

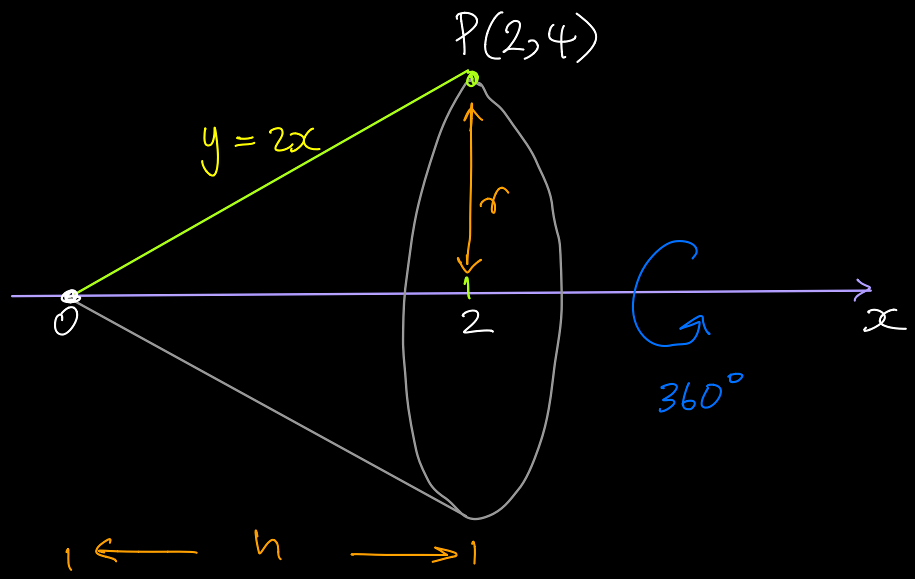
$= 2 \times 2$

$= 4$

Solⁿ 7(b) One approach to solving this problem is to compute the volumes generated when line and the curve are rotated about the x-axis. Then, subtract the results to find the solid volume of revolution

APPROACH I

So, for the rotation of the line $Y = 2x$, we get:



CONE

$r = 4$
$h = 2$

Now, $V_2 = \int_0^2 \pi Y^2 \, dx$

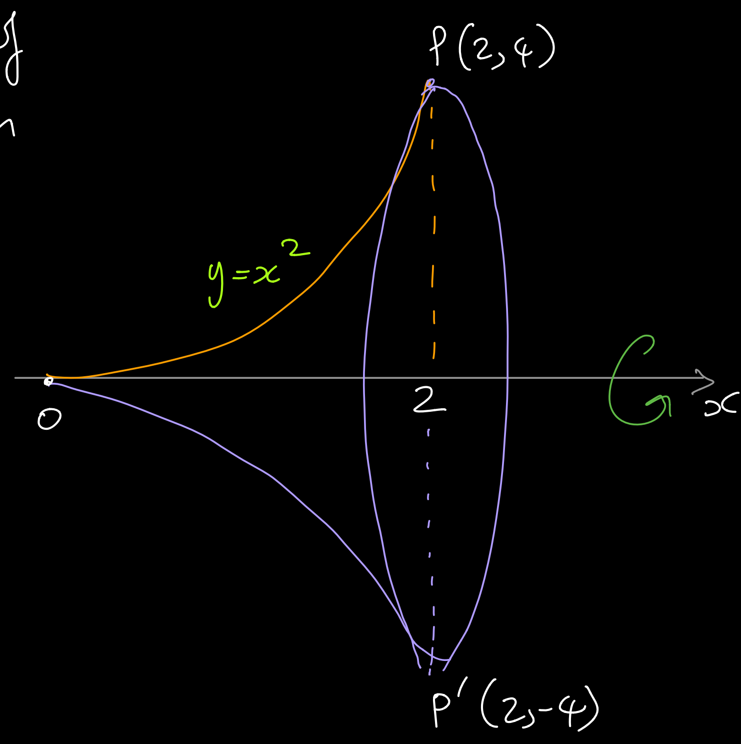
$= \frac{1}{3}\pi r^2 h$ i.e. volume of the cone

$= \frac{1}{3}\pi \times 4^2 \times 2$

$= \frac{32\pi}{3}$ cubic units

In addition, the rotation of the curve $y = x^2$ results in the sketch shown.

Using principles of calculus, the resulting volume is:



$$V_1 = \int_0^2 \pi y^2 \, dx$$

$$= \int_0^2 \pi x^4 \, dx$$

$$= \left[\pi \frac{x^5}{5} \right]_0^2$$

$$= \frac{\pi}{5}(32 - 0)$$

$$= \frac{32}{5}\pi \quad \text{cubic units}$$

Hence, the req. volume is the difference

$$V_T = V_2 - V_1$$

$$= \frac{32}{3}\pi - \frac{32}{5}\pi$$

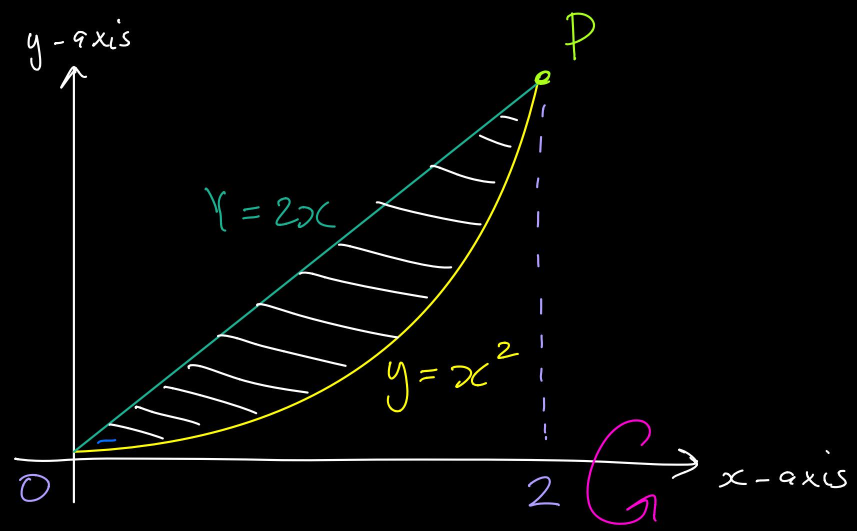
$$= \frac{64}{15}\pi \quad \text{cubic units}$$

This is where we paused in class ...

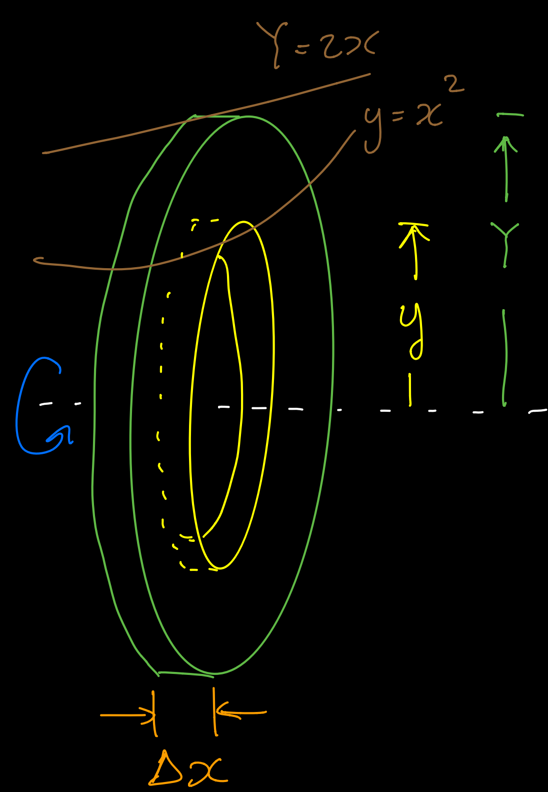
APPROACH II

Now, let's re-solve this problem using a single integration. Recall the construct below:

Sol^n: 7(b) (redo)



The element of volume formed when $Y = 2x$ and $y = x^2$ are simultaneously rotated 360° about the x-axis is



$$\Delta V = \Delta V_2 - \Delta V_1$$

$$= \pi Y^2 \Delta x - \pi y^2 \Delta x$$

$$= \pi (Y - y)^2 \Delta x$$

So, in the limit, $\Delta x \to dx$ and $\Delta V \to dV$.

Hence, by integration,

$$\int_0^{V_T} dV = \int_0^2 \pi (Y - y)^2 \, dx$$

$$\Rightarrow \quad V_T = \int_0^2 \pi \left(2x - x^2\right)^2 dx$$

perfect square expansion

$$= \int_0^2 \pi \left(4x^2 - 4x^3 + x^4\right) dx$$

$$= \pi \left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5} \right]_0^2$$

$$= \pi \left\{ \left(\frac{32}{3} - 16 + \frac{32}{5} \right) - \left(0 - 0 + 0 \right) \right\}$$

$$= \pi \left(\frac{160 - 240 + 96}{15} \right)$$

$$= \frac{16}{15}\pi \text{ cubic units}$$

ENFIN