UNIT 1: PURE MATHEMATICS

Algebra, Geometry and Calculus

(Extract by Boszik from the CAPE 2013 syllabus for Hampton School students; internal use only)

M1: BASIC ALGEBRA AND FUNCTIONS

(a) Reasoning and Logic

1. *Identify simple and compound propositions;*
2. *Establish the truth value of compound statements using truth tables;*
3. *State the converse, contrapositive and inverse of a conditional (implication) statement;*
4. *Determine whether two statements are logically equivalent;*

(b) The Real Number System – *R*

1. *Perform binary operations;*
2. *use the concepts of identity, closure, inverse, commutativity, associativity, distributivity of addition and multiplication and other simple binary operations;*
3. *perform operations involving surds;*
4. construct simple proofs, specifically direct proofs, or proof by the use of counter examples;
5. use the summation notation ($\sum$);
6. establish simple proofs by using the principle of mathematical induction.

(c) Algebraic Operations

1. *apply the Remainder Theorem;*
2. use the Factor Theorem to find factors and to evaluate unknown coefficients;
3. extract all factors of $a^n - b^n$ for positive integers $n \leq 6$;
4. use the concept of identity of polynomial expressions.

(d) Exponential and Logarithmic Functions

1. *define an exponential function* $y = a^x$ *for* $a \in \boldsymbol{R}$;
2. *sketch the graph of* $y = a^x$;
3. *define a logarithm function as the inverse of an exponential function*;
4. *define the exponential function* $y = e^x$ and its inverse $y = \ln x$, where $\ln x \equiv \log_e x$;
5. *use the fact that* $y = \ln x \Leftrightarrow x = e^y$;
6. *simplify expressions by using the laws of logarithms*, such as:
   - (i) $\log (PQ) = \log P + \log Q$,
   - (ii) $\log(P/Q) = \log P - \log Q$,
   - (iii) $\log P^a = a \log P$;
7. *use logarithms to solve equations of the form* $a^x = b$;
8. solve problems involving changing of the base of a logarithm.

(e) Functions

1. *define mathematically the terms: function, domain, range, one-to-one function (injective function), onto function (surjective function), one-to-one and onto function (bijective function), composition and inverse functions;*
2. *prove whether or not a given function is one-to-one or onto and if its inverse exists;*
3. *use the fact that a function may be defined as a set of ordered pairs*;

4. *use* the fact that if g is the inverse function of f, then $f[g(x)] = x$, for all x, *in* the domain of g;

5. illustrate by means of graphs, the relationship between the function $y = f(x)$ given in graphical form and $y=|f(x)|$ and the inverse of $f(x)$, that is $y = f^{-1}(x)$.

(f) The Modulus *Function*

1. *define the modulus function*

$$|x| = \begin{cases} +x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases};$$

2. use the properties:

   (a) $|x|$ is the positive square root of x^2;

   (b) $|x| < |y|$ if, and only if, $x^2 < y^2$;

   (c) $|x| < y \Leftrightarrow$ *iff* $-y < x < y$;

   (d) $|x + y| \leq |y| + |y|$, ("triangular law").

3. *solve equations and inequalities involving the modulus functions, using algebraic and graphical methods.*

(g) *Cubic Functions and* Equations

use the relationship between the *sums of the roots, the products of the roots, the sum of the product of the roots pair-wise* and the coefficients of $ax^3 + bx^2 + cx + d = 0$.

M2: *TRIGONOMETRY*, GEOMETRY & VECTORS

(a) Trigonometric Functions, Identities and Equations (all angles in radians u.o.s.)

1. *use the compound-angle formulae for* sin $(A \pm B)$, cos $(A \pm B)$ *and* tan $(A \pm B)$;

2. *use the reciprocal functions* sec x, cosec x *and* cot x;

3. *Derive identities for the following:*

   (a) sin kA, cos kA, tan kA, for $k \in \mathbf{Q}$;

   (c) $\tan^2 x$, $\cot^2 x$, $\sec^2 x$, and $\text{cosec}^2 x$;

   (e) sin A ± sin B, cos A ± cos B.

4. *further prove identities of Specific Objective 3;*

5. express $a\cos\theta + b\sin\theta$ in the form $r\cos(\theta \pm \alpha)$ and $r\sin(\theta \pm \alpha)$, where r is positive, $0 < \alpha < \pi/2$

6. find the general solution of equations of the forms:

   (a) $\sin k\theta = s$,

   (b) $\cos k\theta = c$,

   (c) $\tan k\theta = t$,

   (d) $a\sin\theta + b\cos\theta = c$, for $a, b, c, k, \in \mathbf{R}$;

7. *find the solutions of the equations in Specific Objective* **6** *above for a given range;*

8. obtain maximum or minimum values of $f(\theta) = a\cos\theta + b\sin\theta$ for $0 \leq \theta \leq 2\pi$.

(b) Co-ordinate Geometry

1. find equations of tangents and normals to circles*;*

2. find the points of intersection of a curve with a straight line;

3. find the points of intersection of two curves;

4. obtain the Cartesian equation of a curve given its parametric representation;

5. obtain the parametric representation of a curve given its Cartesian equation;

6. *determine the loci of points satisfying given properties.*

(c) Vectors

1. express a vector in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or *x**i**+y**j**+z**k** where **i**, **j** and **k** are unit vectors in the x-, y- and z-axis, respectively;*
2. define equality of two vectors;
3. add and subtract vectors;
4. multiply a vector by a scalar quantity;
5. *derive and use unit vectors, position vectors and displacement vectors;*
6. find the magnitude and direction of a vector;
7. find the angle between two given vectors using scalar product;
8. *find the equation of a line in (i) vector form **p**=**a**+λ**d**, (ii) parametric form with λ, or (iii) Cartesian form, given a point A on the line and a vector **d** parallel to the line*; or given 2 points on the line.
9. *determine whether two lines are parallel, intersecting, or skewed*;
10. find the *equation of the plane*, in (i) standard vector form **r.n** = **a.n** = d or (ii) its cartesian form a*x***i** + by**j** + cz**k** = *d*, given a point A on the plane and the normal to the plane **n** = a**i** + b**j** + c**k**.

M3: CALCULUS I

(a) Limits

1. use graphs to determine the continuity and discontinuity of functions;
2. describe the behaviour of a function *f(x)* as *x* gets arbitrarily close to some given fixed number, using a descriptive approach;
3. Use the limit notation $\lim_{x \to a} f(x) = L$, $f(x) \to L$ as $x \to a$;
4. use the simple limit theorems: If $\lim_{x \to a} f(x) = F$, $\lim_{x \to a} g(x) = G$ and k is a constant, then $\lim_{x \to a} kf(x) = kF$, $\lim_{x \to a} f(x)g(x) = FG$, $\lim_{x \to a} \{f(x) + g(x)\} = F + G$, and, provided $G \neq 0$, $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$;
5. *use limit theorems in simple problems*;
6. *use the fact that* $\lim_{x \to 0} \frac{\sin x}{x} = 1$, *demonstrated by a geometric approach*;
7. *identify the point(s) for which a function is (un)defined*;
8. identify the points for which a function is continuous;
9. *identify the point(s) where a function is discontinuous*;
10. *use the concept of left-handed or right-handed limit, and continuity.*

(b) Differentiation I

1. *define the derivative of a function at a point as a limit;*
2. *differentiate, from first principles, such functions as:*
   - (a) *f(x)* = *k* where $k \in \mathbf{R}$,
   - (b) $f(x) = x^n$, $n \in \{\pm 1, \pm 1/2, \pm 2, \pm 3\}$,
   - (c) *f*(*x*) = sin *x*,
   - (d) *f*(*x*) = cos *x*.
3. use the *sum*, product and quotient rules for differentiation;
4. differentiate *sums*, products & quotients of
   - (a) polynomials,
   - (b) trigonometric functions;

5. apply the chain rule in the differentiation
   (a) *composite functions (substitution)*,
   (b) *functions given by parametric equations*;
6. *solve problems involving rates of change*;
7. use the sign of the derivative to investigate where a function is increasing or decreasing;
8. *apply the concept of stationary (critical) points*;
9. calculate second derivatives;
10. interpret the significance of the sign of the second derivative;
11. use the sign of the second derivative to determine the nature of stationary points;
12. *sketch graphs of polynomials, rational functions and trigonometric functions using the features of the function and its first and second derivatives (including vertical and horizontal asymptotes)*;
13. describe the behaviour of such graphs for large values of the independent variable;
14. obtain equations of tangents and normals to curves.

(c) Integration I

1. *recognize integration as the reverse process of differentiation*;
2. demonstrate an understanding of the indefinite integral and the use of the integration notation $\int f(x)dx$;
3. show that the indefinite integral represents a family of functions which differ by constants;
4. demonstrate use of the following integration theorems:
   (a) $\int cf(x)dx = c\int f(x)dx$, c is a constant,
   (b) $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$;
5. find:
   (a) indefinite integrals using integration theorems,
   (b) integrals of polynomial functions,
   (c) integrals of simple trigonometric functions;
6. *integrate œ using substitution*;
7. use the results:
   (a) $\int_a^b f(x)dx = \int_{ta}^{tb} f(t)dt$,
   (b) $\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$,
   (b) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$,
   (c) $\int_a^b f(x)dx = F(b) - F(a)$, where $F'(x) = f(x)$.
8. apply *integration* to:
   (a) finding areas under the curve,
   (b) finding areas between two curves,
   (c) *finding volumes of revolution by rotating regions about both the x- and y-axes*;
9. given a rate of change with or without initial boundary conditions;
   (a) formulate a differential equation of the form $y' = f(x)$ or $y'' = f(x)$ where *f* is a polynomial or a trigonometric function.
   (b) solve the resulting differential equation in (a) above and interpret the solution where applicable.