ADVANCED LEVEL MATHEMATICS

BÖ§ZïK Inc.™ © R2S Nexus

INTEGRATION

Our First 80 Tutorial Questions

Dear Students,

Welcome to the foundation topic of **integral calculus**. The enclosed document contains:

(a) fundamental mathematical reviews of definitions and properties

(b) several problems pertaining to your first calculus topic of Integration

The reviews highlight the significant achievements of how Mathematicians used "infinitely small change" to arrive at a most fundamental theorem of integral calculus. Summarised also are all the formulae and properties of the *integration operator* that you need at this level. The generic problem sets were compiled for our tutorial sessions and they sufficiently cover your syllabus requirements.

Please take this drill seriously and feel free to alert me (as early as you can) if you have any challenges, modifications or corrections as you revise and work ahead of our regularly scheduled class sessions. I will develop the solution key shortly and we will revise these topics in the days leading up to your external examinations. Thank you.

Respectfully,

G. David Boswell

(Please Turn Over)

ADVANCED LEVEL MATHEMATICS

INTEGRAL CALCULUS

PART I ~ REVIEW

Contents

(Please Turn Over)

Contents

(Please Turn Over)

SET A BASICS OF INTEGRATION

A.1 Review of Concepts and Definitions of the Integration (Antiderivative) Operator

(a) From Small Changes, Derivatives and Limits to the Integration Operator

As discussed in class, a "small change" or increment tends to (or approaches) an instantaneous value as the increment tends to zero. With 'tends to' denoted by the arrow '$\to$', we then write $\Delta x \to dx$ as $\Delta x \to 0$, $\Delta y \to dy$ as $\Delta y \to 0$, etc. This *small change concept* was used to elegantly define the differentiation process earlier. That is, under small changes, $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$ such that $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$ *if the increments are infinitesimally small*. This allows us write the definition $\frac{dy}{dx} \triangleq \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$. It also furnished the very special derivative or *gradient function* operator, namely $\frac{d}{dx}$ that pivots on the use the theory of <u>limits</u>!

Similarly, the idea of 'putting back small pieces together or integrating' via *summation* has also being developed from <u>limits</u>! From above, $\Delta y \approx \frac{dy}{dx}\Delta x$. If we sum the LHS and RHS increments, then we will obtain the *approximation* $\sum \Delta y \approx \sum \frac{dy}{dx}\Delta x$. Furthermore, in the limit as $\Delta y \to 0$ and $\Delta x \to 0$, these aggregates become *exact* as:

$$\lim_{\Delta y \to 0}\left(\sum \Delta y\right) = \lim_{\Delta x \to 0}\left(\sum \frac{dy}{dx}\Delta x\right).$$

It is this equation that we rewrite in a compact form using what we now refer to as the *integration operator*. That is $\int dy = \int \frac{dy}{dx}dx$ which, reduces to $y = \int \frac{dy}{dx}dx$. By inspecting this result, we carefully observe that the integral of the derivative of a function results in the function itself. Hence, it is for this reason that the definition of integration is basically the process of finding an *antiderivative*.

Finally, the integration operator[1] is $\int ...dx$. We will see later that the independent variable *x* can be changed, depending on the nature of the problem.

---

[1] The integration operator is an elongated "S" from the Latin word "summa", a noun meaning "sum"

(b) The Indefinite Integral

If $y = f(x)$ is a continuous function, then its indefinite integral that defines a "family of curves" is

$$\int f(x)dx = F(x) + C$$

Here, $f(x)$ is the **integrand**, $F(x)$ is the **integral** and C is the constant of integration. It determines the ***family of curves*** of the solution (see class notes).

(c) The Definite Integral

If $y = f(x)$ is a continuous function on the closed interval $[a,b]$, then its definite integral that defines a numerical solution is

$$\int_a^b f(x)dx = [F(x) + C]_a^b$$
$$= F(b) - F(a)$$

Here, the integration constant need not be known as it vanished during limits evaluation. **This is the Fundamental Theorem of Integral Calculus.**[2]

(d) Selected Properties of the Definite Integral **(Extremely important to know!)**

If $y = f(x)$ is a continuous function on the closed interval $[a,b]$, then the following are properties and geometric interpretations of definite integrals.

$\int_a^b f(x)dx = \int_{ta}^{tb} f(t)dt$	change of variable from *x* to *t*
$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$	shift *f*(*x*) <u>and</u> the limits by *c* units[3]
$\int_0^a f(x)dx = \int_0^a f(a-x)dx$	shift *f* by *a* units then *y*-axis reflection[4]
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$	splits the integral in segments *iff* $a < c < b$
$\int_a^b f(x)dx = -\int_b^a f(x)dx$	interchanging upper and lower limits

[2] F(x) is continuous on the closed interval [a, b] and differentiable on the open interval (a, b)

[3] If c > 0, then f(x-c) is a shift of y = f(x) by c units to the left. The converse holds true.

[4] This equation corrects that which is present in the CAPE 2013 Pure Maths syllabus

(e) Selected Properties of all Integrals during Algebraic Operations

If $f(x)$ and $g(x)$ are continuous functions of x, and c,α,β are arbitrary constants, then

$$\int cf\,dx = c\int f\,dx$$ handling constant *multiplier*

$$\int f \pm g\,dx = \int f\,dx \pm \int g\,dx$$ distributive property

$$\int \alpha f \pm \beta g\,dx = \alpha\int f\,dx \pm \beta\int g\,dx$$ linearity property

Note that the linearity property is a direct consequence (**corollary**) of the previous 2 properties.

Furthermore, no strict "product" and "quotient" rules are applicable in integration. So, please observe the following cautions:

$$\int fg\,dx \neq \int f\,dx \times \int g\,dx$$ (caution!)

$$\int \frac{f}{g}\,dx \neq \frac{\int f\,dx}{\int g\,dx}$$ (caution!)

Finally, to conclude our basic review, we have seen that the process of Integration is "continuous summation" while Differentiation is "continuous gradient search". Today, thanks to the related works of genius by Leibniz and Newton, the latter of whom made the statement in a letter, "If I have seen further it is by standing on the shoulders of Giants" that made direct reference to the work of other pioneers in Mathematics and Science such as Archimedes, Pythagoras, Euclid, Aristotle, Galileo, Kepler, and Descartes.

We now turn our attention to some elementary applications[5] of integration, namely, finding enclosed areas and volumes of revolutions.

[5] A vast number of other applications exist in all fields of study and vector calculus is one such exciting subfield.

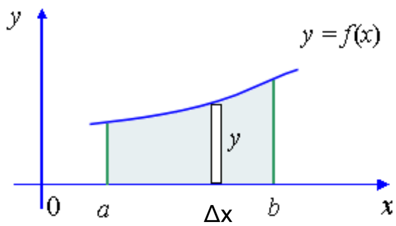
A.2 Review of Some Elementary Applications of Integration

(a) Area Enclosed between a Segment of a Curve and an Axis

If $y = f(x)$ is a continuous function on the closed interval $[a,b]$, then the area enclosed between its curve and the x-axis can be found by an *integration technique*. The process involves slicing the desired 2D region under consideration into thin rectangular strips of equal thickness such that the '*element of area*' is $\Delta A = \text{height} \times \text{width} = y\Delta x$. By summing all the elements of area and then taking the limit as their thicknesses becomes infinitely small ($\Delta x \to dx$), we get

$$\text{Approx. Area, } A \approx \sum_{x=a}^{b} \Delta A$$

$$\approx \sum_{x=a}^{b} y\Delta x$$



$$\text{So, Exact Area, } A = \lim_{\Delta x \to 0}\left(\sum_{x=a}^{b} y\Delta x\right)$$

$$\triangleq \int_a^b y\,dx \qquad \text{square units}$$

Comments

An area is a scalar quantity (always positive). While the desired area is between a curve and the x-axis, an integration process may return a negative result and this is logically corrected using the following heuristics:

IF $y = f(x) > 0$ for all x on the interval $a \le x \le b$ THEN

The region of interest is <u>above</u> the x-axis such that: Area, $A = +\int_a^b y\,dx$

IF $y = f(x) < 0$ for all x on the interval $a \le x \le b$ THEN

The region of interest is <u>below</u> the x-axis such that: Area, $A = -\int_a^b y\,dx$

Question for the Reader

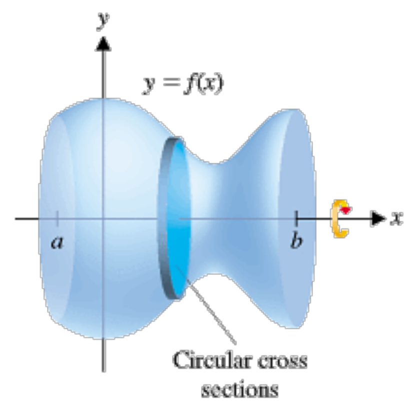
What formulation should be used when the enclosure is w.r.t. the y-axis? (What will be the element of area, lower limit, and upper limit?)

(b) Volume generated by Revolving a Segment of a Curve around an Axis

If $y = f(x)$ is a continuous function on the closed interval $[a,b]$, then the volume generated by rotating its curve 360^0 about the x-axis can be found by an *integration technique*. The process involves slicing the desired 3D region under consideration into thin cylindrical disks of equal thickness such that the '*element of volume*' is $\Delta V = (\text{cross section area})\times(\text{thickness}) = A\Delta x = \pi r^2 \Delta x = \pi y^2 \Delta x$. By summing all the elements of volume and then taking the limit as their thicknesses becomes infinitely small ($\Delta x \to dx$), we get

$$\text{Approx. Volume, } V \approx \sum_{x=a}^{b} \Delta V$$
$$\approx \sum_{x=a}^{b} \pi y^2 \Delta x$$

$$\text{So, the exact volume, } V = \lim_{\Delta x \to 0}\left(\sum_{x=a}^{b} \pi y^2 \Delta x\right)$$
$$\triangleq \int_a^b \pi y^2\, dx \qquad \text{cubic units}$$



Comments

When considering a rotation by θ degrees of the curve about the x-axis, then the total or fractional volume generated is:

$$V = \frac{\theta}{360^\circ}\int_a^b \pi y^2\, dx \qquad \text{cubic units}$$

Better yet, if the angle of rotation was specified in radians (or θ^c), then the volume generated is:

$$V = \frac{\theta^c}{2\pi}\int_a^b \pi y^2\, dx$$
$$= \frac{\theta^c}{2}\int_a^b y^2\, dx \qquad \text{cubic units}$$

Question for the Reader

What formulation should be used when the rotation is w.r.t. the y-axis? (What will be the element of volume, lower limit, and upper limit?)

SET B TABLE OF FORMULAE

B.1 Table of Basic Integrals

Category	Integrand, $f(x)$	Integral, $F(x)=\int f(x)dx$
Constants	a	$ax+C$
Polynomial[6]	ax^n	$\frac{ax^{n+1}}{n+1}+C$, provided that $n\neq -1$
Trigonometric	$\sin x$	$-\cos x+C$
	$\cos x$	$\sin x+C$
Exponential[7]	e^x	e^x+C

B.2 Table of primal results of "Integration by Substitution"

Category	Integrand, $f(x)$	Integral, $F(x)=\int f(x)dx$
Polynomial	$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1}+C$, $n\neq -1$
Trigonometric[8]	$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)+C$
	$\cos(ax+b)$	$-\frac{1}{a}\sin(ax+b)+C$
Exponential[9]	e^{ax+b}	$\frac{1}{a}e^{ax+b}+C$

[6] When n = -1, the result involves Naperian Logarithms. This will be handled in Unit 2

[7] We will not be solving any problems involving the integral of exponential functions in Unit 1

[8] These are the forms needed to solve the integral of "simple" trigonometric functions at this level

[9] Quite simple, but not required in your preparations for Unit 1 examinations

SET C INDEFINITE INTEGRALS

C.1 Find the following integrals of *polynomials*

1. $\int x\,dx$

2. $\int 5\,dx$

3. $\int dx$ Hint: this is interpreted as $\int dx = \int 1\,dx = \int x^0\,dx$

4. $\int \left(x^2 + x + 1\right)dx$

5. $\int \left(-12x^2 - 25\right)dx$

6. $\int 8x^2(3x+7)\,dx$ Hint: first expand the brackets

7. $\int x^{-2}\,dx$

8. $5\int x^{-6}\,dx$

9. $\int \sqrt{x}\,dx$ Hint: rewrite as $\int x^{\frac{1}{2}}\,dx$

10. $\int x^{-\frac{2}{3}}\,dx$

11. $\int \left(x + \frac{3}{\sqrt[4]{x^3}}\right)dx$ Hint: rewrite as $\int x + 3x^{-\frac{3}{4}}\,dx$

12. $\int \frac{12}{\sqrt{x}}\,dx$

13. $\int (x+4)(2-5x)\,dx$

14. $\int (4x+2)^3\,dx$

15. $\int \left(6x^2 - 3\right)^2 dx$

16. $\int \frac{3x^2 - 5x + 2}{x^5}\,dx$

17. $\int x^{-4}\left(x^2 - 3x - 2\right)dx$

18. $\int x^4\left(\frac{3}{x^2} - \frac{1}{2x^4} + \frac{9}{x^6}\right)dx$

C.2 Find the following integrals of *trigonometric functions*

19. $\int \sin x \, dx$

20. $\int \cos x \, dx$

21. $\int (1 - \cos x + \sin x) dx$

22. $\int (6\sin x - 5\cos x + 10) dx$

23. $\int (3 - 2\cos x) dx$

24. $\int \sin 3x - \cos 5x \, dx$

25. $\int 6\sin(3x - 2) dx$

26. $\int \sin\left(\frac{12x + 2}{4}\right) dx$

27. $\int \cos\left(\frac{x}{8} + 1\right) dx$

28. $\int \sin^2 x \, dx$ Hint: express $\sin^2 x$ in terms of $\cos 2x$

29. $\int \cos^2 x \, dx$ Hint: express $\cos^2 x$ in terms of $\cos 2x$

30. $\int \frac{\cot x}{\csc x} dx$

C.3 Find the following integrals of *functions with independent variable other than x*

31. $\int 5 \, dt$

32. $\int u^2 \, du$

33. $\int (3t^2 - 4t + 10) dt$

34. $\int (5\sin u) du$

35. $\int 3\cos(2\theta + 1) d\theta$

36. $\int \sin(2\pi\theta + 15^\circ) d\theta$

SET D DEFINITE INTEGRALS

D.1 Areas using definite integrals involving *polynomials*

Evaluate the following:

37. $\int_{2}^{10} dx$

38. $\int_{0}^{1} 2x+4\,dx$

39. $\int_{-2}^{3} 9x^2\,dx$

40. $\int_{-1}^{2} y^2\,dy$

41. $\int_{9}^{25} \frac{1}{\sqrt{u}}\,du$

42. Discuss why **(i)** $\int_{-4}^{6} \frac{1}{x}\,dx$ and **(ii)** $\int_{2}^{5} \frac{10}{x-3}\,dx$ not will yield any useful results.

43. This problem demonstrates the need to know the shape of a graph before it is safe to find "area under the curve." Now, sketch the graph of $y = x^2 - 5x + 6$ over the [0,4] and compute the area enclosed between this curve and the x-axis from:

- **i.** $x = 0$ to $x = 2$
- **ii.** $x = 2$ to $x = 3$
- **iii.** $x = 3$ to $x = 4$
- **iv.** $x = 0$ to $x = 4$ using the results of **(i)** to **(iii)**

44. Consider graphs of the curve C given by $y = x^3$ and the line L given by $y = x$.

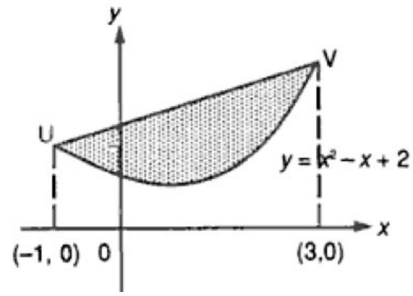
- **i.** Find all points of intersections of these 2 graphs
- **ii.** Determine the stationary point(s) of $y = x^3$ and if any found, show whether it's a turning point (maxima or minima) or a point of inflexion
- **iii.** Sketch diagram showing **(i)**
- **iv.** Carefully find the total area enclosed between C and L.

45. Find the area between the line $y = 2x + 1$ and **(i)** the x-axis from $x = 0$ to $x = 2$, as well as **(ii)** the y-axis from $y = 1$ to $y = 5$. (Hint: See notes, sketch diagrams.)

46. Consider the region trapped by the curve $y = x^2 - x + 2$ and the line UV as shown. Find

i. the coordinates of the points U and V

ii. the area of the shaded region



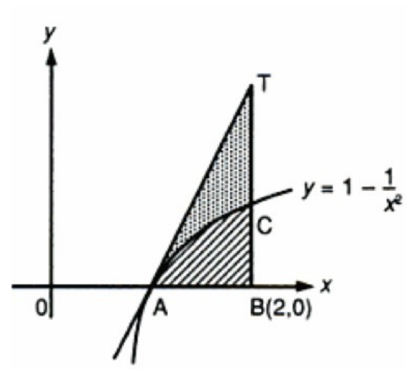
47. Consider a portion of the curve $y = 1 - \frac{1}{x^2}$. Find

i. the coordinates of the point A where the curve meets the x-axis

ii. the equation of the tangent to the curve that passes through A

iii. the area of the shaded region ABC

iv. the area of the shaded region ACT



D.2 Areas using definite integrals involving *trigonometric functions*

Evaluate the following:

48. $\int_0^{\pi} \sin x\,dx$ (note that the angles are in radians)

49. $\int_{-\pi/2}^{3\pi/2} \cos x\,dx$

50. $\int_{0^\circ}^{10^\circ} \cos x\,dx$

51. $\int_0^{\pi/8} (\sin x + \cos 2x)\,dx$

52. This problem demonstrates the need to know the shape of a graph before it is safe to find "area under the curve." Compute the area enclosed by

i. $y = \sin x$ and the x-axis over the interval $[0, 2\pi]$ (See notes!)

ii. $y = \cos x$ and the x-axis over the interval $[0, 2\pi]$ (sketch first)

53. **(Bonus)** Find the smallest value of the constant a such that

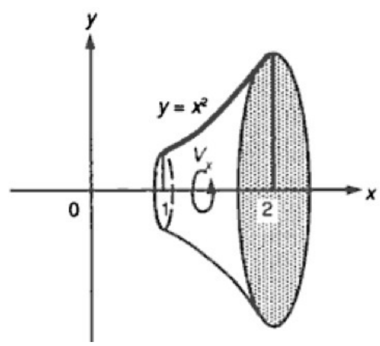
$$\int_0^{a} \sin x\,dx = \int_0^{a} \cos x\,dx$$

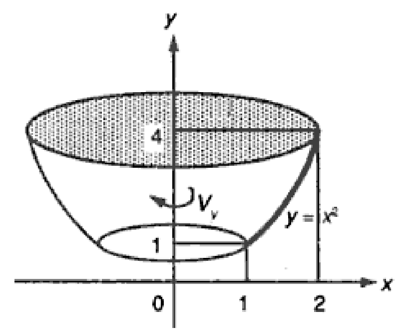
D.3 Volumes using definite integrals involving ***polynomials***

54. Find the volume of the solid of revolution produced when the line $y = 4x^3$ is spun 360° about the x-axis on the interval $[0,2]$.

55. Find the volume generated when $y = \sqrt{x}$ is rotated about the x-axis from $x = 2$ to $x = 3$.

56. The portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ is rotated through 360° about the **(i)** x-axis and **(ii)** y-axis. Leaving π in your answers, find the volumes created.



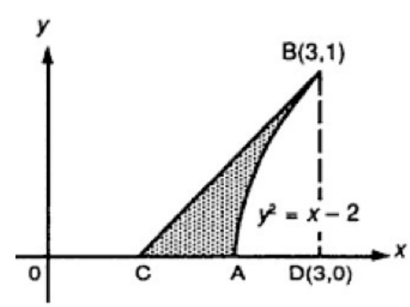


57. The figure below shows a curve $y^2 = x - 2$ with tangent of slope of 0.5 at the point B(3,1). Find

i. the equation of the line BC,

ii. the coordinates of the point C, and

iii. the volume generated when the shaded region revolved about the x-axis



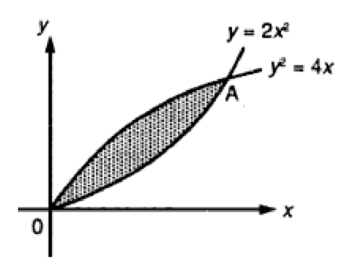
58. The curves $y^2 = 4x$ and $y = 2x^2$ intersect at the origin and the point A. Find

i. the coordinates of point A, and

ii. the volume generated when the region bounded by the 2 curves is rotated about the

   i. x-axis

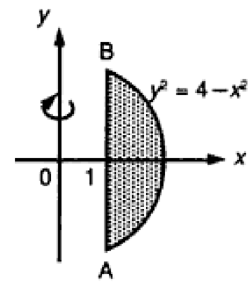
   ii. y-axis



59. The curve $y^2 = 4 - x^2$ and the line $x = 1$ intersect at the points A and B. Find

i. the coordinates of the points A and B, and

ii. the volume generated when the region bounded by the curve and the line is rotated about the y-axis



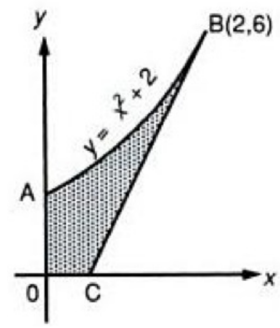
60. **(Bonus)** A part of the curve $y = x^2 + 2$ where B is the point (2,6) is shown in the diagram. The tangent to the curve at B meets the x-axis as C. Find

i. the coordinates of the y-intercept A,

ii. the equation of the tangent BC,

iii. the coordinates of the point C, and

iv. the volume generated when the shaded region is completely rotated about the x-axis



D.4 Volumes using definite integrals involving *trigonometric functions*

61. The portion of the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated through 360° about the x-axis. Find the volume of the resulting solid of revolution. (Hint: use the identity $\cos 2A \equiv 1 - 2\sin^2 A$ as needed)

62. A portion of the curve $y = \cos x$ between $x = \frac{\pi}{2}$ and $x = \pi$ was rotated through 360° about the y-axis. Find the volume of the resulting solid of revolution. (Hint: use the identity $\cos 2A \equiv 2\cos^2 A - 1$ as needed)

(Please Turn Over)

D.5 Miscellaneous Problems involving use of the *Properties of Integrals*

63. If $\int_{-4}^{4} f(x)\,dx = 20$ and a product is defined by $k = \left(\int_{-4}^{0} f(x)\,dx\right) \times \left(\int_{0}^{4} f(x)\,dx\right)$, find the value of k when $f(x)$ is

i. an odd continuous function and

ii. an even continuous function.

64. If $\int_{2}^{4} f(x)\,dx = 6$, what is the value of $\int_{2}^{4} \left(1 - 2f(x)\right) dx$?

65. If $\int_{0}^{3} f(x)\,dx = -7$, what is the value of $\int_{0}^{3} \left(\frac{5}{2} - \frac{f(3-x)}{2}\right) dx$?

66. If $\int_{7}^{21} f(x+8)\,dx = \int_{c}^{d} f(t)\,dt$ under a change of variable, what are the values of the new limits of integration *d* and *c*?

67. If $\int_{0.2}^{0.5} \sin x\,dx = p$, deduce, in terms of *p*, the value of:

i. $\int_{0.5}^{0.2} 4p \sin x\,dx$

ii. $\left(\int_{0}^{0.3} \sin(x + 0.2)\,dx\right)^2$

iii. $\int_{0.1}^{0.25} \left(\sin 2t\right) dt$ (hint: use a change of variable from *x* to *t*)

(Please Turn Over)

SET E MODELING & SOLVING DIFFERENTIAL EQUATIONS

E.1 Modeling problem scenarios

68. At constant temperature, the variation of the pressure P of a certain gas at time t is inversely proportional to its rate of change of volume V at the same time. If the rate of reduction of the pressure of the gas is 40 m^3s^{-1}, find the differential equation expressing the rate at which the volume is changing (use k = constant of proportionality).[10]

69. The rate at which the population P of an urban area is increasing was found to be proportional to the current population. Using k as a constant of proportionality, write down a first order differential equation of this social behaviour.

70. If a cylindrical tank of radius r is being emptied of its liquid content and the height of the remaining liquid at time t is $h(t)$, what is the relationship between the rate of change of the remaining volume $V(t)$ and rate of change of height?

71. Given that acceleration, a is the rate of change of velocity, v and velocity is the rate of change of displacement, x. If an objects acceleration was found to be opposite and proportional to its displacement, develop the differential equation.

72. The rate of change of a property y of a mobile phone signal was found to vary inversely to the cube of the distance x from the phone's antenna to the source of transmission. If k is the proportionality constant, find the differential equation.

73. A gas cylinder a food processing facility contained an initial mass of propane, m_0. If the rate at which the propane is used is directly proportional to the remaining mass, $m(t)$ at time t, develop a differential equation for this system.

(**Suggestion**: Please check CAPE 2013 Specimen Paper (Unit 1, Paper 1, Question 45) for another classic Q&A of this type and also your textbook for several others.)

---

[10] This problem is a corrected statement

E.2 Solve the following basic differential equations

(For these type of problems, integrate both sides with w.r.t. the independent variable)

74. Given that $\frac{dy}{dx} = 3$ with $y(1) = -12$, find its solution $y(x)$.

75. Provided that $\frac{dy}{dx} = 2x + 1$ with $y(0) = 5$, find $y(x)$.

76. $y'(x) = 3x^2 - 4x + 4$ is a first order differential equation whose solution is satisfied by $y(1) = 0$. Use direct integration to find its solution $y(x)$.

77. Given the second order differential equation as $\frac{d^2y}{dx^2} = 6x - 2$ with $y'(0) = 3$ and $y(0) = -7$, find its solution.

78. Given the second order differential equation[11] as $y''(x) = 12x^2 + 8$ with $y(0) = 10$ and $y(1) = 2$, find the solution of y in terms of x.

79. A curve has turning point $(1,-1)$ and gradient function $\frac{dy}{dx} = 6x^2 + ax - 12$. Find the value of a and the equation of the curve.

80. Given that $\frac{d^2y}{dt^2} = 3 - kt$ where k is a constant, if

i. $\frac{dy}{dt} = -6$ when $t = -1$ and 9 when $t = 2$, find the value of k

ii. $y = -6.5$ when $t = 1$, find the equation of y in terms of t (i.e $y(t)$)

(**Suggestion**: Please check CAPE 2013 Specimen Papers (Unit 1) for another Q&A of this type and also your textbook for several others.)

[11] This problem is an update to a previous question with a typographical error in this problem set

SET F WEB LINKS (ANIMATIONS, ETC)

1. “A brief history of Integration,” http://integrals.wolfram.com/about/history/

2. Wolfram Mathematica™ - Online Integrator web applet or browser plugin (limited but useful) - http://integrals.wolfram.com/index.jsp (The full Mathematica software package for Mac and Windows is unlimited)

3. Geogebra™ - Dynamic mathematics & science for learning and teaching (software for Windows, Mac, Linux, etc) - http://www.geogebra.org/cms/en/

4. A Visual of “Volume of Revolution,”http://upload.wikimedia.org/wikipedia/commons/e/e7/Rotationskoerper_animation.gif

5. “Mathematics Animations - Derivatives,” (this animation and commentary is an extract), http://www.youtube.com/watch?v=ZL9Eyt7tDuw

6. “Mathematics Animations - Integration,” (this animation and commentary is an extract), http://www.youtube.com/watch?v=OQvFdRGffag

7. **Take a Break** ~ Manipulating Lines, Surfaces, Volumes, Time and Change (the abstraction and applications of Mathematics) in a randomly selected 3-D Animation found on YouTube, http://www.youtube.com/watch?v=7bSCpeIO6ys … **Enjoy!**

- ENFIN -