ADVANCED LEVEL MATHEMATICS

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DIFFERENTIATION

Our First 57 Tutorial Questions

The enclosed *draft* document contains **57 problems** and basic mathematical **reviews** pertaining to your first calculus topic of Differentiation. These generic problem sets were compiled for our tutorial sessions.

Please take this drill seriously and feel free to alert me (as early as you can) if you have any challenges, modifications or corrections as you revise and work ahead of our regularly scheduled class sessions.

The content is as follows.

Contents

Thanks.

(Please Turn Over)

SET A DIFFERENTIATION OF POLYNOMIALS

Review ~ If a and n are real constants, then $\frac{d}{dx}\left(ax^n\right) = nax^{n-1}, \quad \forall n \in \mathbb{R}$

Problems ~ Find the *first derivative* of the following functions w.r.t. the independent variable. (Note the use of the different but equivalent notations.)

1. $y = x^2 + 2x + 3$; $\frac{dy}{dx} = ?$

2. $y = 3x^5 - 4x^3 + 9x - 6$; $\frac{dy}{dx} = ?$

3. $f(x) = x^9 - 5x^8 + x + 12$; $f'(x) = ?$

4. $y(x) = \frac{1}{4}x^8 - \frac{1}{6}x^6 - x + 2$; $\frac{dy}{dx} = ?$

5. $y = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{\sqrt{x}}$; $y'(x) = ?$

6. $y = \frac{3}{x} - \frac{1}{x^2} + \frac{2}{3x^3}$; $y' = ?$

7. $y = \sqrt{x^3} + \frac{1}{\sqrt{x^3}}$; $\frac{dy}{dx} = ?$

8. $y = 2\sqrt{x^3} + \frac{4}{\sqrt{x}} - \sqrt{2}$

9. $y = -\frac{x^2}{16} + \frac{2}{x} - x^{\frac{3}{2}} + \frac{1}{3x^2} + \frac{x}{3}$

10. $f(x) = \frac{3}{x+5}$; $\frac{df}{dx} = ?$

11. $y = \frac{x+1}{x-2}$

12. $y = \frac{2x-3}{5x+4}$

13. $y = \frac{4}{x^2+2}$

14. $y = \frac{x^2+2x+1}{3}$

SET B DIFFERENTIATION USING THE "CHAIN RULE"

Review ~ If y and x are functions such that $y = f(u)$ and $x = g(u)$, then by the "Chain Rule",

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. This can also be written as $\frac{dy}{dx} = \frac{dy}{du} \Big/ \frac{dx}{du}$. Furthermore, note that $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$.

Problems ~ With an appropriate substitution, use the Chain Rule to find the *first derivative* of the following functions w.r.t. the independent variable.

15. $y = \sin 3x$; $u = 3x$

16. $y = \cos 8x$; $u = 8x$

17. $y = \sin(13x + 2)$; $u = 13x + 2$

18. $y = 4\cos(8x^2 + 4x + 1)$; $u = 8x^2 + 4x + 1$

19. $\sqrt{\sin x}$

20. $\sqrt[3]{\cos x}$

21. $\sqrt[3]{x^2 + 1}$

22. $\frac{1}{3x + 2}$, $x \neq \frac{2}{3}$

23. $(9x^3 + 6x^2 - 10x + 1)^5$

24. $\left(x + \frac{1}{x}\right)^3$

25. $\left(\sqrt{x} + 2\right)^4$

26. $(4x^2 + 2x)^2$

27. $\left(3x^2 + \frac{1}{x}\right)^9$

28. $\frac{1}{(1 - x^2)^4}$

(By the end of this exercise, you should be able to solve many of these problems by *inspection*. (Class discussion: what is meant by an "*inner derivative*"?))

SET C DIFFERENTIATION USING THE "PRODUCT RULE"

Review ~ If $u(x)$ and $v(x)$ are differentiable functions of x, then, the derivative of the product $y(x)=u(x)v(x)$ can be found using the formula

$$\frac{dy}{dx}=\frac{d(uv)}{dx}$$
$$=u\frac{dv}{dx}+v\frac{du}{dx}$$

This is often times abbreviated as $d(uv)=udv+vdu$ or $(uv)'=uv'+vu'$

Problems ~ With appropriate substitutions and without expanding, use the Product Rule to find the *first derivative* of the following functions w.r.t. the independent variable.

29. $y=(2x+1)(3x-2)$; $\frac{dy}{dx}=?$

30. $y=50(5-x^2)(2x+7)$; $y'=?$

31. $y=(x^2+5)(1-3x)$

32. $(x^2+1)(1-x^3)$ hint: with $y=f(x)$ then find $\frac{dy}{dx}$

33. $(x+1)\sin x$

34. $(x^2+3x-1)\cos x$

35. $y=x(x^3-3x)$

36. $y=(\sqrt{x+1})(\sqrt{x-1})$

37. $y=\frac{2x-3}{5x+4}$ hint: $\frac{1}{5x+4}=(5x+4)^{-1}$

38. $y=(4x^2+x-1)\left(\frac{1}{x}-\frac{1}{x^2}\right)$

SET D DIFFERENTIATION USING THE "QUOTIENT RULE"

Review ~ If $u(x)$ and $v(x)$ are differentiable functions with respect to x, and importantly, $v(x) \neq 0$, then the derivative of the quotient $y(x) = \dfrac{u(x)}{v(x)}$ can be found using the formula

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

This is often times abbreviated as $d(u/v) = \dfrac{vdu - udv}{v^2}$ or $(u/v)' = \dfrac{vu' - uv'}{v^2}$.

Problems ~ With <u>appropriate</u> substitutions, use the Quotient Rule to find the *first derivative* of the following functions w.r.t. the independent variable. Also, state the condition for which the function and its derivative exist.

39. $y = \dfrac{x^2 + 1}{x - 1}$; $\dfrac{dy}{dx} = ?$

40. $y = \dfrac{x + 1}{x - 2}$; $\dfrac{dy}{dx} = ?$

41. $y = \dfrac{x^2 - 9x + 1}{3x + 1}$; $y'(x) = ?$

42. $y = \dfrac{(x - 2)^3}{x^2}$; $y' = ?$

43. $g(x) = \dfrac{3x^4 - 100}{2e^x}$; $\dfrac{dg}{dx} = ?$

44. $y = \tan x$ hint: $\tan x = \dfrac{\sin x}{\cos x}$

45. $y = \cot x$ hint: $\tan x = \dfrac{\cos x}{\sin x}$

46. $y = \sec x$ hint: ?

47. $y = \csc x$ hint: ?

SET E INTRO TO THE APPLICATIONS OF DIFFERENTIATION

Recap / Review ~

First Principles: $f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$

Product Rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ i.e., $(uv)' = uv' + vu'$

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, \quad v \neq 0$ i.e., $(u/v)' = \frac{vu' - uv'}{v^2}, \quad v \neq 0$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

“Reciprocal”: $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$ such that $\frac{dy}{dx} = \frac{dy}{du} \div \frac{dx}{du}$

Small Change: Using the first principle definition of the derivative of a function, we write $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ such that when Δx is “small,” $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$.

Hence, $\Delta y \approx \frac{dy}{dx}\Delta x$. So, overall, given that $x^{new} = x^{old} + \Delta x$, the update is

$$\begin{aligned} y^{new} &= y^{old} + \Delta y \\ &\approx y^{old} + \frac{dy}{dx}\Delta x \end{aligned}$$

Problems ~ Solve the following problems, some of which are *parametric* and *rate of change* equations.

48. Find the coordinates of the point on the curve $y = -6x^2 + 2x + 5$ where the gradient or slope vanished! (hint: find $\frac{dy}{dx}$ and solve $\frac{dy}{dx} = 0$ for x).

49. A circle C is defined by the pair of parametric equations given as $x = 5 + 3\cos\theta$ and $y = -8 + 3\sin\theta$. Find the following:

(a) $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$

(b) Hence or otherwise, $\frac{dy}{dx}$

(c) Coordinates of the point $P(x, y)$ on the circle where $\theta = \frac{\pi^c}{3}$

(d) Gradient at the $P(x, y)$ on C where $\theta = \frac{\pi^c}{2}$

50. Given that $g(t) = \frac{1}{t^2}$ and $f(t) = \frac{2t-3}{2}$, then

(a) Find $\frac{dg}{dt}$ and $\frac{df}{dt}$

(b) Hence or otherwise, $\frac{df}{dg}$ in terms of t (hint: use the chain rule)

(c) Using the result in (b), find an expression for $\frac{dg}{df}$

51. Provided that a curve is described by the equation $y = u^3 - 4u^2 + 6u$ where $u = x^2 + 1$, then, find the value of $y'(2)$. (Hint: $y'(2) = \left.\frac{dy}{dx}\right|_{x=2}$).

52. Use the "chain rule" to find an expression for $\frac{dy}{dx}$ given that $y = 5t^2 + 3t - 1$ where $t = 4x + 10$.

53. A curve is parametrically defined by $y = 5t^2$ and $x = 3t + 1$. Find the gradient of the point on the curve where $t = 2$.

54. If the cartesian coordinates of a certain trace are uniquely determined at any time by the equations $x = \sqrt{t}$ and $y = \sin t$, find an expression in terms of t for $\frac{dy}{dx}$.

55. **Bonus** ~ A large helium filled balloon at an exciting birthday party is losing gas at a constant rate of $3.0\ cm^3 s^{-1}$. A (nerdy) observer, instead of dancing, wishes to find the rate at which the radius is decreasing when its surface area is $400\pi\ cm^2$.

(a) What is the value of the rate of change in volume, $\frac{dV}{dt}$?

(b) Compute the radius of the balloon when its surface area is $400\pi\ cm^2$.

(c) What is the rate at which the radius is decreasing when its surface area has reached the value stated above?

(Hint: Assume the balloon is spherical of radius r such that such that its volume is $V = \frac{4}{3}\pi r^3$ and surface area is $A = 4\pi r^2$)

56. **Bonus** ~ If water is being pumped from the base of a large upright cylindrical reservoir at a rate of 100 m^3/min, find the rate at which the height of the water in the reservoir is changing if its diameter is 10 m long.

(Hints: (i) Write down the formula for the volume of a cylinder, (ii) note the constants, and (iii) differentiate both sides with respect to the variable, time t)

57. **Bonus** ~ Given that the square root function is expressed by the equation $y = \sqrt{x}$, using "small changes" principle, estimate the value of $\sqrt{16.005}$.

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